



Research paper

Do all dual matrices have dual Moore–Penrose generalized inverses ?

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ABSTRACT

This paper investigates the question of whether all dual matrices have dual Moore–Penrose generalized inverses. It shows that there are uncountably many dual matrices that do not have them. The proof is constructive and results in the construction of large sets of matrices whose members do not have such an inverse. Various types of generalized inverses of dual matrices are discussed, and the necessary and sufficient conditions for a dual matrix to be a Moore–Penrose generalized inverse of another dual matrix are provided. Necessary and sufficient conditions for other types of generalized inverses of dual matrices are also provided. A necessary condition, which can be easily computed, for a matrix to be a {1,2}-generalized inverse or a Moore–Penrose Inverse of a dual matrix is given. Dual matrices that have no generalized inverses arise in practical situations. This is shown by considering a simple example in kinematics. The paper points out that in other areas of science and engineering where dual matrices are also commonly used, formulations and computations that involve their generalized inverses need to be handled with considerable care. This is because unlike generalized inverses of ordinary matrices, generalized inverses of dual matrices do not always exist.

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1. Introduction

Dual matrices have been a topic of recent interest and over the last three decades or so they have gained considerable prominence because of their applicability to various areas of engineering like the kinematic analysis and synthesis of spatial mechanisms [1–5], and robotics [6–10]. The increasing applicability of dual matrices to various areas of science and engineering has also sparked increased interest in the linear algebra aspects related to their usage [11,12]. Angeles [13] develops the use of dual generalized inverses in kinematic synthesis. The authors have already discussed some theoretical issues and contributed to the computation of dual generalized inverses with different procedures and concise formulas [1,3,12,14].

One of the motivations behind this paper and our interest in dual matrices, is the capability of dual numbers to capture both translations and rotations and to develop tools to further expand the already numerous applications of the algebra of dual matrices in different areas. For instance, in kinematics, thanks to the Principle of Transference [15,16], many problems

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can be initially formulated assuming a spherical motion and then, after dualization of equations, these are generalized for a screw motion.

The pioneering work of engineering applications of dual algebra is due to Denavit [17], Dimentberg [18], Keler [19], Beyer [20], Yang [21], Yang and Freudenstein [22], Soni and Harrisberger [23]. The concept of the widely adopted Denavit-Hartenberg parameters has likely been inspired by the geometric interpretation of a dual angle.

In the following we present a short list of contributions with solutions of kinematic problems adopting dual algebra as the main theoretical tool.

A common application is the extraction of rigid body screw parameters by means of dual numbers e.g. Ravani and Ge [24,25], Teu and Kim [26], Condurache and Burlacu [27–31]. More recently, Perez-Gracia and Thomas [32], apply Cayley's factorization to obtain the screw parameters of a rigid body motion as dual Euler parameters. Condurache and Burlacu [33,34] address a new approach for solving the $AX = XB$ sensor calibration problem in robotics. The numerical data are from noisy measurements and an SVD filtering procedure is applied.

An iterative numerical procedure for the kinematic analysis of closed loop linkages is due to Cheng and Thompson [35]. This is one of the first contributions where the solution of the system of dual equations was achieved without splitting the real and dual parts. For this purpose, the Ch programming language was developed where the dual number and the dual arrays were a built-in native data types. Cohen and Shoham [36] exploited the features of dual functions to automatically find first and second order derivatives of constraint equations. These features are particularly useful to obtain the Jacobian matrix in analytical form.

Regarding kinematic synthesis based on dual algebra, Perez-Gracia and McCarthy [37,38] deduced the solution for the three-position synthesis of a Bennett linkage. The screws of the relative displacements, the nodal line of the cylindroid are expressed in dual notation. Perez-Gracia and McCarthy [39] formulated dual quaternion design equations for robot synthesis and applied them to the solution of the Revolute-Prismatic-Revolute-Prismatic (RPRP) serial chain with 5 imposed complete positions and 4 translations. Bai and Angeles [40,41] adopt dual algebra for the formulation of the spatial Burmester problem and for the synthesis of RCCC¹ linkages whose coupler visits four given poses. Design equations for RC and CC dyads are deduced. The design equations are presented in matrix notation, thus well suited for the theory herein developed. Pennestri et al. [14] proposed an extension of the dual SVD decomposition and a compact formula for computing the dual Moore-Penrose generalized inverse of dual matrices. The results have been applied toward the kinematic synthesis of the RCCC function generator.

Formulations based on dual numbers for the dynamic analysis of mechanical systems have been also proposed by distinguished investigators. In his textbook Fischer [5] provides several examples of dual algebra applications to the kinematic, static, and dynamic analysis of closed loop spatial linkages. The dual algebra equations are in such cases solved by separating the real and the dual parts.

Pennock and Oncu [42] describe the dynamic state of motion of a rigid body by the dual version of the Euler equation. Brodsky and Shoham [43–45] introduced a dual inertia operator and systematically applied it for the derivation of dual momentum and dual energy to obtain a dual version of the Newton-Euler and Lagrange equations. Cohen and Shoham [46] extended the application of the dual inertia operator to hyper-dual numbers. The hyper-dual momentum was defined and an application of the concept to a two cylindrical four degrees-of-freedom robot was offered. Yang and Pennock [47] developed an analytic technique based on dual numbers to derive first order kinematics features (velocity and screw axis) for the motion of a robotic manipulator.

Regarding the solution of systems of dual equations, a least-squares approach system has been recently offered by Belzile and Angeles [48]. In particular, they proposed a dual version of the Householder algorithm for QR decomposition of dual matrices. In the real case, the Moore-Penrose generalized inverse is associated with the minimum norm, least-squares solution of a linear system of equations. However, for the dual case there are difficulties in defining the norm of a dual vector; for instance, a dual vector may represent a rigid body displacement. Martinez and Duffy [49], for a set of finite rigid bodies displacements, proposed metric space structures compliant with the axioms of a metric space. Kazerounian and Rastegar [50] addressed the problem of frame invariant distance norms. Park [51] observes that any distance metric in SE(3) will depend on the length scale. Gupta [52], for body displacements represented by 4×4 real matrices, investigates measures of rotational and translational errors. In particular, he discusses how a simple matrix difference could lead to a meaningful positional error norm. The theoretical difficulties that are encountered in defining distance metrics for general screw displacements are explained well by Angeles [53].

In this paper we investigate the fundamental question of whether every dual matrix has a {1,2,3,4}-dual generalized inverse, also often called a Moore-Penrose dual generalized inverse—MPDGI, for short. The analog of this question for ordinary matrices would be: does every matrix have a Moore-Penrose inverse? As is well known, the Moore-Penrose inverse of a matrix has numerous areas of application, for example, in statistics [54–55], multibody dynamics, linear algebra aspects of generalized inverses [56–58], the theory of constrained motion [59–61], and precision control of dynamical systems [62–64]. And since this question when posed for ordinary matrices has the answer that every matrix has a unique Moore-Penrose inverse, it might be intuitively thought that the same would indeed be true for dual matrices. Considering the vast potential for using dual matrices in kinematics and dynamics, as well as in other fields of application, it is therefore of interest to find

¹ Revolute-Cylindrical-Cylindrical-Cylindrical kinematic pairs

out if results about ordinary matrices can also be extended to dual matrices. This paper shows that this cannot be done and that there are uncountably many dual matrices that have no Moore–Penrose dual generalized inverses (MPDGI). The proof of this result is done in a constructive manner, showing how to construct large sets of matrices that have no MPDGI.

Section 2 of the paper deals with a constructive proof that all dual matrices do not have MPDGIs. Symbolic examples are provided along the way to illustrate the proof. This section also provides necessary and sufficient conditions for the MPDGI of a dual matrix to exist. The existence of other types of generalized inverses, like the {1,2}-inverse, of dual matrices are also given here. In Section 3.1, a practical example from kinematics, an area of engineering importance where dual matrices are commonly used, is taken, showing that one can obtain both in simulations and in experiments dual matrices that have no MPDGIs. Computations based on the assumption of the existence of MPDGI of dual matrices can therefore lead to erroneous results. Dual matrices are often used in the solution of inverse problems that rely on the use of measured data. In Section 3.2 the inverse problem of estimating a dual rotation matrix from measurement data is considered in some detail. Noisy measurement data on the positions of points on a rigid flat panel before and after rotation are used to estimate the dual rotation matrix. The pitfalls of trying to compute the Moore–Penrose generalized inverse of a dual matrix in solving this inverse kinematical problem, without vetting it to ascertain whether its dual generalized inverse exists, are illustrated. Section 4 provides the main conclusions of the paper.

2. MPDGI of dual matrices

Lemma 1. Consider the dual matrix $\hat{A} = A + \varepsilon B$ in which both A and B are m by n matrices, and ε is a basis element of the (hypercomplex) dual algebra with $\varepsilon^2 = 0$. If an n by m dual matrix $\hat{G} = G + \varepsilon R$ is a {1,2,3,4}-generalized inverse of the matrix, then $G = A^+$, where A^+ denotes the Moore–Penrose(MP) inverse of the m by n matrix A . Therefore, the {1,2,3,4}-dual generalized inverse $\hat{G} = A^+ + \varepsilon R$, if it exists, for some suitable matrix R .

Proof. For \hat{G} to be the {1,2,3,4}-dual generalized inverse of \hat{A} , the matrix \hat{G} must satisfy the following MP conditions.

$$(1) \hat{A}\hat{G}\hat{A} = \hat{A}, (2) \hat{G}\hat{A}\hat{G} = \hat{G}, (3) \hat{A}\hat{G} = (\hat{A}\hat{G})^T, \text{ and } (4) \hat{G}\hat{A} = (\hat{G}\hat{A})^T. \tag{1}$$

under the proviso that $\varepsilon^2 = 0$.

We now look at each of the above-mentioned four dual MP conditions given in Eq. (1).

(1) The first condition requires that

$$[(A + \varepsilon B)(G + \varepsilon R)(A + \varepsilon B)]|_{\varepsilon^2=0} = A + \varepsilon B, \tag{2}$$

which simplifies to

$$AGA + \varepsilon(BGA + ARA + AGB) = A + \varepsilon B. \tag{3}$$

Hence,

$$AGA = A, \tag{4}$$

and

$$B = BGA + ARA + AGB. \tag{5}$$

(2) The second condition requires that

$$[(G + \varepsilon R)(A + \varepsilon B)(G + \varepsilon R)]|_{\varepsilon^2=0} = G + \varepsilon R, \tag{6}$$

which simplifies to

$$GAG + \varepsilon(RAG + GBG + GAR) = G + \varepsilon R.$$

Hence,

$$GAG = G \tag{7}$$

and

$$R = RAG + GBG + GAR. \tag{8}$$

(3) The third condition requires that

$$[(A + \varepsilon B)(G + \varepsilon R)]|_{\varepsilon^2=0} = [(A + \varepsilon B)(G + \varepsilon R)]^T|_{\varepsilon^2=0} \tag{9}$$

which simplifies to the two conditions

$$AG = (AG)^T \tag{10}$$

and

$$(BG + AR) = (BG + AR)^T \tag{11}$$

(4) The fourth condition requires that

$$[(G + \varepsilon R)(A + \varepsilon B)]|_{\varepsilon^2=0} = [(G + \varepsilon R)(A + \varepsilon B)]^T|_{\varepsilon^2=0} \quad (12)$$

which simplifies to the two conditions

$$GA = (GA)^T \quad (13)$$

and

$$(RA + GB) = (RA + GB)^T. \quad (14)$$

Hence for the four conditions given in Eq. set (1) to be satisfied, Eqs. (4),(7),(10), and (13), must each be satisfied. But the set of Eqs. (4),(7),(10), and (13) are the Moore–Penrose conditions for the matrix G to be the Moore–Penrose inverse of the matrix A !

In fact, matrices G that satisfy only Eq. (4) for a given matrix A are referred to in the literature as $\{1\}$ -generalized inverses of A , and often denoted by $A^{\{1\}}$; those that satisfy only Eqs. (4) and (7) are called the $\{1,2\}$ -generalized inverses of A and are referred to as $A^{\{1,2\}}$; and those that satisfy only Eqs. (4),(7),(10), are called $\{1,2,3\}$ -generalized inverses of A and denoted by $A^{\{1,2,3\}}$. The (unique) matrix that satisfies all 4 of these equations Eqs. (4),(7),(10), and (13) is referred to as the $\{1,2,3,4\}$ -generalized inverse of A , or simply as the Moore–Penrose generalized inverse of A , and it is denoted by $A^{\{1,2,3,4\}}$ or A^+ , for short.

Hence, if $\hat{G} = G + \varepsilon R$ is a $\{1,2,3,4\}$ -dual generalized inverse of $\hat{A} = A + \varepsilon B$, then $G = A^+$. That is, the MPGI of the dual matrix \hat{A} must have the form $\hat{G} = A^+ + \varepsilon R$, for some suitable matrix R . \square

Remark 1. Lemma 1 can be checked in a simple and intuitive manner. Consider the dual matrix $\hat{A} = A + \varepsilon B$, and its $\{1,2,3,4\}$ -dual generalized inverse $\hat{G} = G + \varepsilon R$. Now $\hat{A}|_{B=0} = A$, which is real, and whose $\{1,2,3,4\}$ -dual inverse, $\hat{G} = A^+ = G$. \square

Lemma 2. As shown in Lemma 1, the $\{1\}$ -generalized inverse of the matrix $\hat{A} = A + \varepsilon B$, which satisfies only the first MP condition, must have the form $\hat{A}^{\{1\}} = A^{\{1\}} + \varepsilon R$, where R is a suitable matrix that satisfies Eq. (5) with $G = A^{\{1\}}$; the $\{1,2\}$ -generalized inverse, which satisfies only the first and second MP conditions, must have the form by $\hat{A}^{\{1,2\}} = A^{\{1,2\}} + \varepsilon R$, where R is a suitable matrix that satisfies Eqs. (5) and (8), with $G = A^{\{1,2\}}$; the $\{1,2,4\}$ -generalized inverse, which satisfies only the first, second, and fourth MP conditions, must have the form $\hat{A}^{\{1,2,4\}} = A^{\{1,2,4\}} + \varepsilon R$ where R satisfies Eqs. (5),(8), and (11) with $G = A^{\{1,2,4\}}$, etc.

Hence the real part, G , of the any type of generalized inverse—like, say, the $\{1,2\}$ - or the $\{1,2,3\}$ -generalized inverse, etc.—of the dual matrix \hat{A} is independent of the matrices B and R which we shall call the dual components of \hat{A} and \hat{G} , respectively. In fact, the real component of any type of generalized inverse of a dual matrix is simply the corresponding type of generalized inverse of its real component. The main difficulty in finding the generalized inverse of a dual matrix then rests in finding the dual component R of the generalized inverse, if it exists. \square

Result 1. : There exist dual matrices which have no $\{1,2,3,4\}$ -dual generalized inverses.

Proof. Consider a given matrix $\hat{A} = A + \varepsilon B$, and assume that its $\{1,2,3,4\}$ -dual generalized inverse (MPDGI) is the matrix $\hat{G} = G + \varepsilon R$. Then, by Lemma 1, $G = A^+$, and $\hat{G} = A^+ + \varepsilon R$. Thus finding the MPDGI matrix, \hat{G} , reduces to finding the proper matrix R so that the four conditions in the Eq. set (1) are satisfied. In this result we construct a set, $S_{\hat{A}}$, of dual matrices \hat{A} , and show that for every member of this set, $S_{\hat{A}}$, there does not exist any matrix R that satisfies all the conditions given in equation set (1); hence, no MPDGI exists for the matrices in this set $S_{\hat{A}}$. The set $S_{\hat{A}}$ contains an uncountably infinite number of matrices.

(a) Construction of the set, $S_{\hat{A}}$, of matrices $\hat{A} = A + \varepsilon B$

Construction of the set of matrices A: Consider any m by n diagonal matrix A with $m \geq n$, that has rank r with $r < n$. We will denote the (i, j) th element of A by $a(i, j)$. The diagonal of the m by n matrix A is comprised of the n elements $a(i, i)$, $i = 1, 2, \dots, n$. We will call A a diagonal matrix for short, if its only nonzero elements lie on its diagonal. Also, since A has rank $r < n$, r of the diagonal elements of A must be non-zero, and the remaining $(n - r)$ of them must be zero. We describe the structure of any diagonal m by n matrix A by using two (complimentary) sets, A_z (or A_{nz}).

We assume that the $(n - r)$ diagonal elements $a(i_k, i_k)$, $k = 1, 2, \dots, (n - r)$, of A are zero. The set of indices i_k of A for which the elements $a(i_k, i_k)$ are zero is denoted by $A_z = \{(i_k | a(i_k, i_k) = 0)\}$. In a similar fashion, we denote the set of indices i_k along the diagonal elements for which $a(j_k, j_k) \neq 0$ by the set $A_{nz} = \{j_k | a(j_k, j_k) \neq 0\}$. The subscript 'nz' suggests 'non-zero,' and the set A_z is the complement of the set A_{nz} . In other words, we look at each element along the diagonal of the matrix A . If that element is zero, then we list its row index i (which is the same as its column index) in the set A_z ; if the element is non-zero, we list its row index i in the set A_{nz} . Knowledge of the set A_z (or A_{nz}) for any m by n matrix diagonal matrix A , describes its structure.

Example: Consider the following two 7 by 5 matrices, A and C, whose elements are zero, except for one or more elements on their diagonals:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & a_{55} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Assume that the elements a_{11} , a_{44} , and a_{55} , have arbitrary nonzero values.

Consider first the matrix A. The matrix does not have full rank; its rank is $r = 2$. Three of the diagonal elements, namely, a_{22} , a_{33} , and a_{55} , are zero. The set A_z therefore contains the row indices of these three zero elements on the diagonal of A. Hence $A_z = \{i_1, i_2, i_3\} = \{2, 3, 5\}$. The set A_{nz} contains the indices of the non-zero diagonal elements of A, so $A_{nz} = \{1, 4\}$ since the diagonal elements a_{11} and a_{44} are non-zero. Note that the set of 7 by 5 diagonal matrices that have $A_z = \{2, 3, 5\}$ or $A_{nz} = \{1, 4\}$ are uncountably infinite since the values of elements a_{11} and a_{44} are arbitrary.

In a similar manner, matrix C has rank $r = 3$. Looking at the elements along its diagonal, the set $C_z = \{2, 3\}$ and the set $C_{nz} = \{1, 4, 5\}$. The sets C_z (or C_{nz}) describe the structure of the diagonal matrix C.

Construction of the set of matrices B for the dual matrix $A + \varepsilon B$: We now select for the m by n diagonal matrix A (which has $m \geq n$ and rank $r < n$) a structure that is described by the known sets A_z and A_{nz} . The only elements that are non-zero in the matrix A are the elements $a(i,i)$, $i \in A_{nz}$, which lie along its diagonal. Corresponding to the structure selected for our matrix A, i.e., corresponding to the elements of set A_z (or A_{nz}) chosen, we now prescribe the structure of the set of B matrices.

To construct the set of B matrices, we start with any arbitrary m by n matrix B and alter its elements (if necessary) so that for $i \in A_z$, one or more of its diagonal elements $b(i, i)$ has an arbitrary nonzero value.

This completes our specification of the set $S_{\hat{A}}$ of dual matrices $\hat{A} = A + \varepsilon B$ that we will consider. In words, given a diagonal matrix A, the manner of creating the corresponding set of B matrices is quite simple. We find the locations along the diagonal of the matrix A at which it has zero elements; these locations comprise the set A_z . At one or more of the corresponding locations in the matrix B we place nonzero elements of arbitrary value. The remaining elements of matrix B are given arbitrary values.

We thus construct our set $S_{\hat{A}}$ of dual matrices $\hat{A} = A + \varepsilon B$, where the matrices A and B are constructed as above.

Example (Continued). Continuing our example, we choose our matrix A to have the same structure as the matrix A given in Eq. (15). Corresponding to this choice for the structure of A, we now construct the set of matrices B. To construct the corresponding set of matrices B, start with any arbitrary 7 by 5 matrix and alter one or more of the diagonal elements of B so that at least one $b(i, i)$ with $i \in A_z = \{2, 3, 5\}$ has a non-zero value. That is, one or more of the elements $b(2, 2)$, $b(3, 3)$, $b(5, 5)$ are given arbitrary, nonzero values. Among many others, we show below just four candidate structures for B.

$$B_1 = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & b_{42} & 0 & b_{44} & b_{45} \\ 0 & 0 & 0 & 0 & 0 \\ b_{61} & 0 & b_{63} & b_{64} & b_{65} \\ b_{71} & 0 & b_{73} & b_{74} & b_{75} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b_{61} & b_{62} & 0 & 0 & 0 \\ 0 & 0 & b_{73} & b_{74} & b_{75} \end{bmatrix}, \tag{16}$$

$$B_3 = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 \\ 0 & 0 & 0 & 0 & b_{55} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} \end{bmatrix}. \tag{17}$$

In the matrices B_1 , B_2 and B_3 all the elements shown by the b 's are nonzero. In the more general matrix B shown in Eq. (17), since $A_z = \{2, 3, 5\}$, at least one of the diagonal elements $b(2, 2)$, $b(3, 3)$, $b(5, 5)$ must have a non-zero (arbitrary)

value. The remainder of the elements of the matrix B can take any arbitrary values. It can be seen that for special (zero) values of the elements of the matrix B we obtain the matrices B_1, B_2 and B_3 .

We thus construct the set of matrices $S_{\hat{A}}$ in which $\hat{A} = A + \varepsilon B$, where A has the same structure as the matrix A that is given in Eq. (15), and B is, say, any one of the four structures shown in Eqs. (16) or (17) whose elements we have described. In a further continuation of this example, we shall show that for all elements of the set $S_{\hat{A}}$ no MPDGI exists.

(b) Non-existence of the $\{1,2,3,4\}$ -dual generalized inverse of the constructed matrices $\hat{A} = A + \varepsilon B$ that belong to the set $S_{\hat{A}}$

In what follows we shall consider the general B matrix (shown in Eq. (17)) with its elements satisfying the condition that at least one $b(i, i), i \in A_z$ is non-zero. We consider the matrix $\hat{A} = A + \varepsilon B$, where the matrices A and B are constructed as discussed above under item (a).

According to Lemma 1, the $\{1,2,3,4\}$ -dual inverse of the matrix \hat{A} is $\hat{G} = A^+ + \varepsilon R$ where A^+ is the Moore-Penrose inverse of A , and R is some suitable n by m matrix. Eq. (5) requires that

$$\Delta := \underbrace{\underbrace{BA^+A}_{S_1} + \underbrace{AA^+B}_{S_2} + \underbrace{ARA}_{S_3}}_{S_{12}} - B = 0. \tag{18}$$

We shall now show that with the set of matrices A and B constructed as above in (a) above, this inequality is impossible to satisfy for any n by m matrix R .

To do this we assemble the first three terms comprising the matrix S on the left hand side of Eq. (18) and show that their difference from B can never be zero. In order to illustrate the structure of the matrices S_1, S_2, S_3 , and S , more concretely, we continue the example at each step below.

(1) The diagonal matrix A^+A is an n by n matrix whose elements are all zero except those (i,i) elements with $i \in A_{nz}$, which are unity. Hence the j th column of the m by n matrix $S_1 := BA^+A$, for $j \in A_{nz}$ is the same as the corresponding column of the matrix B ; for $j \in A_z$, the j th column of S_1 is zero. We continue our example to illustrate the structure of S_1 .

Example (Continued). The matrices A^+A and AA^+ are respectively 5 by 5, and 7 by 7, respectively. Each of these square matrices has all its elements zero, except those on the diagonal whose indices belong to the set $A_{nz} = \{1, 4\}$. Hence, except for the (1,1) and the (4,4) elements which are unity, all the other elements of both these matrices are zero. The matrix B has at least one of its elements $b(j, j) \neq 0$ for $j \in A_z$. Thus there is at least one non-zero element among the elements $b(2, 2), b(3, 3), b(5, 5)$.

The matrix S_1 is then obtained as

$$S_1 = BA^+A = \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} \end{bmatrix}}_B \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A^+A} = \begin{bmatrix} b_{11} & 0 & 0 & b_{14} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 \\ b_{31} & 0 & 0 & b_{34} & 0 \\ b_{41} & 0 & 0 & b_{44} & 0 \\ b_{51} & 0 & 0 & b_{54} & 0 \\ b_{61} & 0 & 0 & b_{64} & 0 \\ b_{71} & 0 & 0 & b_{74} & 0 \end{bmatrix}. \tag{19}$$

The other elements of B are arbitrary. Note that the j th column of S_1 is zero for $j \in A_z$.

(2) The matrix AA^+ is an m by m matrix all of whose elements are zero, except those on the diagonal with indices that belong to the set A_{nz} , which are unity. Hence the i th row of the matrix $S_2 := AA^+B$, for $i \in A_{nz}$, is the same as the corresponding row of the matrix B ; for $i \in A_z$, the i th row of S_2 is zero. The other rows of S_2 are all zero. See Eq. (20) below to get a picture of the resulting structure of S_2 .

Example. (Continued):

$$S_2 = AA^+B = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{AA^+} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

Note that the i th row of S_2 in Eq. (20) is zero for $i \in A_z$.

- (3) When $i \in A_z$, the i th row of S_2 is zero; when $i \in A_z$, then the i th column of S_1 is zero. Hence in their sum $S_{12} := S_1 + S_2$, the elements $s_{12}(i, i)$ are zero for $i \in A_z$.
- (4) Consider now an n by m matrix R and the n by n matrix product ARA . Noting that A is a diagonal matrix, the i th row of the matrix AR for $i \in A_z$ is zero. Also, the i th row of the matrix AR is non-zero, in general, for $i \in A_{nz}$. The remaining $(m - r)$ rows of AR are zero. These $(m - r)$ rows that are zero in the matrix AR will continue to be zero in the matrix $S_3 := ARA$. In addition, the j th column of S_3 , for $j \in A_z$, is also zero. Thus the diagonal elements $s_3(i, i)$ are zero for $i \in A_z$.

Example (Continued). The matrix S_3 in our example is given by

$$\begin{aligned}
 S_3 = ARA &= \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} & r_{17} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} & r_{27} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} & r_{37} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} & r_{47} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} & r_{57} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} r_{11}a_{11}^2 & 0 & 0 & r_{14}a_{11}a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r_{41}a_{11}a_{44} & 0 & 0 & r_{44}a_{44}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}
 \end{aligned}$$

As seen above, for $i \in A_z = \{2, 3, 5\}$, the (i, i) elements of S_3 are all zero.

- (5) Summing the three matrices S_1, S_2 and S_3 we find that the j th column, for $j \in A_z$ of the sum S , is zero, except for those elements in the column's i th row, where $i \in A_{nz}$. These elements in the i th row (with $i \in A_{nz}$) of the j th column ($j \in A_z$) which are non-zero are the corresponding (ij) elements of the matrix B . Most importantly, the (i, i) elements of $S = S_{12} + S_3$ are all zero for $i \in A_z$ since these elements are zero both in S_{12} and in S_3 .

Example (Continued). The matrix $S = S_{12} + S_3$ in our example is obtained as

$$S = S_1 + S_2 + S_3 = \begin{bmatrix} r_{11}a_{11}^2 + 2b_{11} & b_{12} & b_{13} & r_{14}a_{11}a_{44} + 2b_{14} & b_{15} \\ b_{21} & 0 & 0 & b_{24} & 0 \\ b_{31} & 0 & 0 & b_{34} & 0 \\ r_{41}a_{11}a_{44} + 2b_{41} & b_{42} & b_{43} & r_{44}a_{44}^2 + 2b_{44} & b_{45} \\ b_{51} & 0 & 0 & b_{54} & 0 \\ b_{61} & 0 & 0 & b_{64} & 0 \\ b_{71} & 0 & 0 & b_{74} & 0 \end{bmatrix}. \tag{22}$$

Again, for $i \in A_z = \{2, 3, 5\}$, the (i, i) elements of S_3 are all zero. Thus, the elements $s(2,2), s(3,3)$ and $s(5,5)$ of the matrix S are each equal to zero, independent of whatever the elements of R may be.

- (6) Hence the matrix $\Delta = S - B$ must have its (i, i) element equal to $-b(i, i)$ for $i \in A_z$. But by construction, (see Part (a) of the proof above) one or more of the elements $b(i, i)$ for $i \in A_z$, is assigned an arbitrary nonzero value. Hence $\Delta \neq 0$. This result states that no matter how the elements of the matrix R are chosen and what they are, the (i, i) elements of the matrix S , for $i \in A_z$, are always zero and $\Delta \neq 0$. Hence the first MP-condition, Eq. (5), is not satisfied, and there exists no MPDGI for the set of dual matrices $S_{\hat{\lambda}}$ constructed in Part (a).

Example (Continued). The matrix $\Delta = S - B$ (see Eq. (18)) is obtained as

$$\Delta = S - B = \begin{bmatrix} r_{11}a_{11}^2 + b_{11} & 0 & 0 & r_{14}a_{11}a_{44} + b_{14} & 0 \\ 0 & -b_{22} & -b_{23} & 0 & -b_{25} \\ 0 & -b_{32} & -b_{33} & 0 & -b_{35} \\ r_{41}a_{11}a_{44} + b_{41} & 0 & 0 & r_{44}a_{44}^2 + b_{44} & 0 \\ 0 & -b_{52} & -b_{53} & 0 & -b_{55} \\ 0 & -b_{62} & -b_{63} & 0 & -b_{65} \\ 0 & -b_{62} & -b_{63} & 0 & -b_{65} \end{bmatrix}. \tag{23}$$

Since at least one of the three elements b_{22}, b_{33}, b_{55} , is nonzero, the matrix $\Delta \neq 0$ no matter what the matrix R is. Hence there is no MP generalized inverse for the matrix $\hat{A} = A + \varepsilon B$ (see Eq. (17)) in which one or more of the diagonal elements

$b(2, 2)$, $b(3, 3)$, $b(5, 5)$ of the matrix B are different from zero. Thus, every matrix belonging to the set $S_{\hat{A}}$ has no $\{1,2,3,4\}$ -dual generalized inverse. \square

Corollary 1. In the result above we have constructed the set $S_{\hat{A}}$ of dual matrices of the form $\hat{A} = A + \varepsilon B$ in which the matrices A and B are m by n with $m \geq n$, A being rank deficient. We could have similarly constructed a set $S_{\hat{A}}$ of matrices with $m \leq n$, with the diagonal matrix A again not having full rank. Our procedure for constructing this new set of dual matrices would be similar. Other than the fact that the number of elements along the diagonal of A would be different and dependent on its dimensions, the procedure and the final result would remain unchanged. Hence, we can construct dual matrices of any dimensions (greater than 2) that would have no $\{1,2,3,4\}$ -dual generalized inverses. \square

Corollary 2. There are an uncountably infinite number of matrices $\hat{A} = A + \varepsilon B$ where A is rank deficient that have no $\{1,2,3,4\}$ -dual generalized inverses. This directly follows from the Result that is proved. \square

Result 2. The necessary and sufficient conditions for the dual m by n matrix $\hat{A} = A + \varepsilon B$ to have the $\{1,2,3,4\}$ -generalized inverse $\hat{A}^{\{1,2,3,4\}} := G + \varepsilon R$ are:

$$1. \quad G = A^+ = A^{\{1,2,3,4\}}, \text{ where we have denoted the MP – inverse of } A \text{ by } A^{\{1,2,3,4\}}, \quad (24)$$

$$2. \quad B = BA^+A + ARA + AA^+B, \text{ and } R = RAA^+ + A^+BA^+ + A^+AR, \text{ and,} \quad (25)$$

$$3. \quad (BA^+ + AR) \text{ and } (RA + A^+B) \text{ are symmetric matrices.} \quad (26)$$

We often denote the generalized dual inverse $\hat{A}^{\{1,2,3,4\}}$ (or, MPDGI) simply by \hat{A}^+ .

Proof. The proof follows directly from Eqs. (4)–(14) in Lemma 1, since $G = A^+$. The two conditions given in Eq. (25) come from the 1st and 2nd Moore–Penrose conditions, respectively; and, the two conditions in Eq. (26) come from the 3rd and 4th Moore–Penrose conditions, respectively. \square

Result 3. Consider the dual matrix $\hat{A} = A + \varepsilon B$ and let its Moore–Penrose generalized dual inverse be $\hat{A}^+ := G + \varepsilon R$. Then the dual matrix $\hat{A}^T := A^T + \varepsilon B^T$ has the Moore–Penrose generalized dual inverse (MPDGI) given by

$$(\hat{A}^T)^+ := (\hat{A}^+)^T = (A^+)^T + \varepsilon R^T = G^T + \varepsilon R^T \quad (27)$$

Proof. Since $G = A^+$ and the real part of $(\hat{A}^T)^+$ is $(A^T)^+ = (A^+)^T = G^T$, it follows that the real part of $(\hat{A}^T)^+$ is G^T . Also, since R is the dual part of \hat{A}^+ , it satisfies Eqs. (24)–(26) Taking the transpose of Eq. (25) we get

$$B^T = B^T G^T A^T + A^T R^T A^T + A^T G^T B^T \text{ and } R^T = R^T A^T G^T + G^T B^T G^T + G^T A^T R^T \quad (28)$$

which are exactly two of the conditions that the matrix R^T must satisfy to be the dual part of the MPDGI of $\hat{A}^T = A^T + \varepsilon B^T$ (see Eq. (25)).

Furthermore, from Eq. (26), the 3rd Moore–Penrose condition for the MPDGI of \hat{A} to be $G + \varepsilon R$ gives

$$(BG + AR)^T = \underbrace{R^T A^T + G^T B^T}_{BG + AR} \quad (29)$$

and the 4th Moore–Penrose condition for the MPDGI of \hat{A} to be $G + \varepsilon R$ gives

$$(RA + GB)^T = \underbrace{B^T G^T + A^T R^T}_{RA + GB} \quad (30)$$

Using the last equality in Eq. (29) and taking its transpose we get

$$(R^T A^T + G^T B^T)^T = R^T A^T + G^T B^T \quad (31)$$

which is the 4th Moore–Penrose condition required to be true for the MPDGI of $\hat{A}^T = A^T + \varepsilon B^T$ to be $G^T + \varepsilon R^T$ (see Eq. (26))!

Similarly, using the last equality in Eq. (30) and taking its transpose we get

$$(B^T G^T + A^T R^T)^T = B^T G^T + A^T R^T \quad (32)$$

which is the 3rd Moore–Penrose condition required to be true for the MPDGI of $\hat{A}^T = A^T + \varepsilon B^T$ to be $G^T + \varepsilon R^T$ (see Eq. (26)).

All the conditions stated in Result 2 above are therefore satisfied by the matrix

$$G^T + \varepsilon R^T.$$

That this result may not be true for other dual generalized inverses, like $\hat{A}^{\{1,2,3\}}$ and $\hat{A}^{\{1,2,4\}}$, is shown later (Corollary 5) \square

Corollary 3. If the dual matrix P does not have an MPDGI, then the dual matrix P^T also does not have an MPDGI.

Proof. Assume that the dual matrix P does not have an MPDGI, and that the dual matrix P^T does have an MPDGI. Since, by assumption, P^T does have an MPDGI, then by Result 3 the transpose of P^T , namely P , must also then have an MPDGI, which contradicts the assumption that P does not have an MPDGI. Hence P^T cannot have an MPDGI. \square

The next result and the following two corollaries relate to $\{1\}$ - and $\{1,2\}$ -dual generalized inverse matrices, denoted by $\hat{A}^{\{1\}}$ and $\hat{A}^{\{1,2\}}$, respectively.

Result 4. The necessary and sufficient conditions for the m by n dual matrix $\hat{A} = A + \varepsilon B$ to have the $\{1\}$ -dual generalized dual inverse $\hat{A}^{\{1\}} = G + \varepsilon R$ are

$$1. \quad G = A^{\{1\}}, \text{ where we have denoted a } \{1\} \text{ - inverse of } A \text{ by } A^{\{1\}}, \text{ i.e., } AA^{\{1\}}A = A. \tag{33}$$

$$2. \quad B = BA^{\{1\}}A + ARA + AA^{\{1\}}B. \tag{34}$$

Proof. Using Eqs. (4) and (5) the result follows. \square

Several similar results as in Result 4 (giving the necessary and sufficient conditions) can be obtained directly by using Eqs. (4)-(14) for the dual inverses $\hat{A}^{\{1,2\}}, \hat{A}^{\{1,2,3\}}, \hat{A}^{\{1,2,4\}}$, etc., which we leave to the reader.

Corollary 4. A necessary (though not sufficient) condition that must be satisfied by a $\{1,2\}$ -dual generalized inverse, $\hat{A}^{\{1,2\}} = G + \varepsilon R$, of the dual matrix $\hat{A} = A + \varepsilon B$ is

$$ARA = -AA^{\{1,2\}}BA^{\{1,2\}}A \tag{35}$$

Proof. From Eqs. (4) and (7) it follows that $G = A^{\{1,2\}}$. Since every $\{1,2\}$ -dual inverse is also a $\{1\}$ -dual inverse, Eq. (5) must also be satisfied by $A^{\{1,2\}}$. On pre-multiplication of Eq. (34) by $AA^{\{1,2\}}$ and post-multiplication by $A^{\{1,2\}}A$ we get

$$\begin{aligned} AA^{\{1,2\}}BA^{\{1,2\}}A &= AA^{\{1,2\}}B \underbrace{AA^{\{1,2\}}A}_{A^{\{1,2\}}} + \underbrace{AA^{\{1,2\}}ARA}_{A} \underbrace{AA^{\{1,2\}}A}_{A} + A \underbrace{AA^{\{1,2\}}AA^{\{1,2\}}}_{A^{\{1,2\}}} BA^{\{1,2\}}A \\ &= AA^{\{1,2\}}BA^{\{1,2\}}A + ARA + AA^{\{1,2\}}BA^{\{1,2\}}A, \end{aligned} \tag{36}$$

which then yields the result. \square

It should be noted that every $\{1,2,3,4\}$ -dual generalized inverse (MPDGI) of the dual matrix $\hat{A} = A + \varepsilon B$ must also be a $\{1,2\}$ -dual generalized inverse. Hence the relation

$$ARA = -AA^+BA^+A \tag{37}$$

also provides a necessary condition that the matrices B and R must satisfy for $\hat{A}^{\{1,2,3,4\}} = A^+ + \varepsilon R$ to be a $\{1,2,3,4\}$ -dual generalized inverse of \hat{A} . Eq. (37) can therefore serve as a simple computational check when computing the MPDGI of a dual matrix. Such a check would of course not be complete, since it is only a necessary condition.

In the next corollary we show that Result 3 is not true for $\{1,2,3\}$ - and $\{1,2,4\}$ -dual generalized inverses.

Corollary 5. If the dual matrix $\hat{A} = A + \varepsilon B$ has a $\{1,2,3\}$ -dual generalized inverse given by $\hat{A}^{\{1,2,3\}} = A^{\{1,2,3\}} + \varepsilon R_1$, then $\hat{A}^T = A^T + \varepsilon B^T$ and $(\hat{A}^T)^{\{1,2,3\}} \neq (A^{\{1,2,3\}})^T + \varepsilon R_1^T$. Likewise, if \hat{A} has a $\{1,2,4\}$ -inverse $\hat{A}^{\{1,2,4\}} = A^{\{1,2,4\}} + \varepsilon R_2$ then $(\hat{A}^T)^{\{1,2,4\}} \neq (A^{\{1,2,4\}})^T + \varepsilon R_2^T$.

Proof. From Eq. (10) in Lemma 1 it is easy to show that the real part of $(\hat{A}^T)^{\{1,2,3\}}$ is not $(A^{\{1,2,3\}})^T$, since this would require $[A^T(A^{\{1,2,3\}})^T]^T = A^T(A^{\{1,2,3\}})^T$ which implies $A^{\{1,2,3\}}A = (A^{\{1,2,3\}}A)^T$, an equality that the matrix $A^{\{1,2,3\}}$ is not obliged to satisfy. Similarly, from Eq. (13) in Lemma 1, in order for the real part of $(\hat{A}^T)^{\{1,2,4\}}$ to equal $(A^{\{1,2,4\}})^T$, we require $[(A^{\{1,2,4\}})^T A^T]^T = (A^{\{1,2,4\}})^T A^T$, which implies that $AA^{\{1,2,4\}} = (AA^{\{1,2,4\}})^T$, an equality that the matrix $A^{\{1,2,4\}}$ is not obliged to satisfy. Hence the result. \square

Result 2 gives the necessary and sufficient conditions for a dual matrix to have a Moore–Penrose dual generalized inverse (MPDGI). The conditions given in the result can be further developed for dual matrices $A + \varepsilon B$ that have the structure

$$P := \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ a_7 & a_8 & a_9 \end{bmatrix}}_A + \varepsilon \underbrace{\begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}}_B \tag{38}$$

in which the real part of the dual matrix P is not zero. Such 3 by 3 dual matrices are of some interest since they can commonly arise in the area of kinematics where dual matrices are often employed, as shown in the following section.

By Result 2, for the matrix P in Eq. (38) to have an MPDGI, which we denote by $P^+ = G + \varepsilon R$, we require that $G = A^+$, which then has the structure

$$G = A^+ = \begin{bmatrix} A_1 & 0 & A_3 \\ A_4 & 0 & A_5 \\ A_7 & 0 & A_9 \end{bmatrix} \tag{39}$$

in which the elements $A_i = A_i(a_1, a_2, a_3, a_7, a_8, a_9), i = 1, 3, 4, 5, 7, 9$, are appropriately obtained by finding the generalized inverse of the ordinary matrix A . To find the MPDGI of the dual matrix P , we are therefore left with only the determination of the matrix

$$R = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}. \tag{40}$$

Corollary 6. A necessary condition for the matrix P in Eq. (38) to have an MPDGI is that the elements of the matrix B satisfy the relations

$$\begin{aligned} d_4 &:= (A_1b_4 + A_4b_5 + A_7b_6)a_1 + (A_3b_4 + A_6b_5 + A_9b_6)a_7 - b_4 = 0 \\ d_5 &:= (A_1b_4 + A_4b_5 + A_7b_6)a_2 + (A_3b_4 + A_6b_5 + A_9b_6)a_8 - b_5 = 0 \\ d_6 &:= (A_1b_4 + A_4b_5 + A_7b_6)a_3 + (A_3b_4 + A_6b_5 + A_9b_6)a_9 - b_6 = 0 \end{aligned} \tag{41}$$

where we have used the notation shown in Eqs. (38) and (39) above.

Note that the relations *only* involve elements of the matrices A, A^+ , and the elements b_4, b_5 , and b_6 , of the matrix B , i.e., only elements of the dual matrix P , and those of the matrix A^+ .

Proof. Eq. (25) requires that

$$BGA + ARA + AGB - B = 0, \tag{42}$$

This simplifies, after some algebra, to the set of linear equations

$$W\mathbf{r} = \begin{bmatrix} a_1^2 & 0 & a_1a_7 & a_1a_2 & 0 & a_2a_7 & a_1a_3 & 0 & a_3a_7 \\ a_1a_2 & 0 & a_1a_8 & a_2^2 & 0 & a_2a_8 & a_3a_2 & 0 & a_3a_8 \\ a_1a_3 & 0 & a_1a_9 & a_3a_2 & 0 & a_2a_9 & a_3^2 & 0 & a_3a_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1a_7 & 0 & a_7^2 & a_1a_8 & 0 & a_8a_7 & a_1a_9 & 0 & a_7a_9 \\ a_2a_7 & 0 & a_8a_7 & a_2a_8 & 0 & a_8^2 & a_2a_9 & 0 & a_8a_9 \\ a_3a_7 & 0 & a_9a_7 & a_3a_8 & 0 & a_9a_8 & a_3a_9 & 0 & a_9^2 \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \end{bmatrix} = \mathbf{d}. \tag{43}$$

From Eq. (43) we observe that: (i) the 9 by 9 matrix W *only* involves element of the matrix A , and (ii) the fourth, fifth, and sixth rows of the matrix W are each zero. Hence, for this set of equations to be consistent, *irrespective of what the elements r_i of the matrix R shown in Eq. (40) are*, we must have $d_4 = d_5 = d_6 = 0$. These three elements of the vector \mathbf{d} in Eq. (43) are obtained algebraically, and they as given in Eq. (41). □

Remark 2. Eq. (41) states the three conditions that the dual matrix P shown in Eq. (38) must satisfy if it to have a Moore–Penrose generalized dual inverse (MPDGI). That these algebraic conditions are necessary, points out that *generically* 3 by 3 matrices with the structure that P has, i.e. dual matrices whose real part has a row (or column) that is zero, do not have MPDGIs. By generically, what is meant here is that among the set of 3 by 3 dual matrices any matrix whose real part has one row (or column) that is zero, *almost always* has no MPDGI.

Another way of saying this is that if the object on which three line-vectors are drawn is a 2-dimensional object, so that the real part of the dual matrix describing these three line-vectors has one row of zeros, then this dual matrix almost always has no MPDGI. We will shortly see this in the next section where this result comes into prominence when we take up an inverse problem in kinematics related to estimation of dual rotation matrices for flat (2-dimensional) solar panels, and then for solar panels that are not flat. □

3. An application from kinematics

This section deals with the estimation of dual rotation matrices from three line-vectors constructed from observations of points on a body before and after rotation. Dual matrix descriptions of such line vectors are commonly used in kinematics. We illustrate the possible problems that a naïve use of dual matrices in inverse problems of kinematics might cause.

3.1. Description of line-vectors

We consider in this subsection the description of line-vectors on the surface of, say, a spacecraft’s solar panel. Such line-vectors are often ‘drawn’ on surfaces whose subsequent motions are required to be simulated and/or monitored as they translate and rotate over time.

We consider six points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, on the surface of the flat solar panel with cartesian coordinates

$$\mathbf{p}_1 = [2 \quad 1 \quad 1]', \mathbf{p}_2 = [3 \quad 1 \quad 3]', \mathbf{p}_3 = [4 \quad 1 \quad -1]' \tag{44}$$

and

$$\mathbf{q}_1 = [3 \quad 1 \quad 2]', \mathbf{q}_2 = [2 \quad 1 \quad 0]', \mathbf{q}_3 = [5 \quad 1 \quad 2]'. \tag{45}$$

The primes indicate transpose of the row vectors. The flat solar panel is in the plane $y = 1$.

Three line-vectors directed from $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, are taken to describe the position of the panel's surface at some time t . Using the coordinates of the points given in Eqs. (44) and (45), the dual matrix that describes these line vectors is

$$P = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -3 & 3 \end{bmatrix}}_W + \varepsilon \begin{bmatrix} 1 & -3 & 3 \\ -1 & 6 & -13 \\ -1 & 1 & -1 \end{bmatrix}. \tag{46}$$

The i th column of the real part of the dual matrix P is $\mathbf{q}_i - \mathbf{p}_i, i = 1, 2, 3$, and the i th column of the dual part of P is $\mathbf{p}_i \times \mathbf{q}_i, i = 1, 2, 3$.

Since the dual matrix P has the structure shown in Eq. (38), we know that it belongs to the generic class of matrices that almost always has no Moore–Penrose generalized dual inverse. To show that this matrix indeed does not have and MPDGI we use Corollary 6 and show that the necessary conditions stated in it are not satisfied.

Denoting the MPDGI of the matrix P by $S + \varepsilon T$ (assuming it exists), we know that $S = W^+$ which computes to

$$S = \begin{bmatrix} \frac{3}{2} & 0 & -\frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}. \tag{47}$$

Knowing the dual matrix P and the matrix S —which is all that is needed—the three relations given in Eq. (41) can be checked. Evaluating d_4, d_5 , and d_6 defined in Eq. (41), we find that

$$d_4 = 0, \quad d_5 = \frac{7}{2}, \quad d_6 = \frac{7}{2}. \tag{48}$$

This shows that no MPDGI of the dual matrix P exists since d_5 and d_6 are not zero. By Corollary 3, P^T also has no MPDGI. This result is corroborated when attempting to compute the MPDGIs of P and P^T .

We now consider the manner in which the lines drawn on the solar panel are displaced by a rotation described by the dual matrix.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \tag{49}$$

which represents a rotation of 90° about the unit vector $\mathbf{v}_0 = [\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0]^T$ passing through the point $w_0 = (0, \quad 0, \quad 1)$.

The final position, P^T after rotation of the three line-vectors represented by the dual matrix P is then given by

$$AP = P'. \tag{50}$$

Using Eqs. (46) and (49), we get

$$P' = \begin{bmatrix} 1.2071 & -2.6123 & 2.6213 \\ -0.2071 & 1.6213 & -1.6213 \\ -0.7071 & 0.7071 & -0.7071 \end{bmatrix} + \varepsilon \begin{bmatrix} -1.0000 & 1.0858 & -4.5858 \\ 1.4142 & -1.3284 & -2.1716 \\ -2.1213 & 7.0711 & -12.0208 \end{bmatrix}. \tag{51}$$

One can confirm by using the simple theory of rotations that the points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, in Eqs. (44) and (45) are taken, after rotation, to the points (primes are used for coordinates after rotation)

$$\mathbf{p}'_1 = [1.5 \quad 1.5 \quad 0.2929]', \mathbf{p}'_2 = [3.4142 \quad 0.5858 \quad -0.4142]', \mathbf{p}'_3 = [1.0858 \quad 3.9142 \quad -1.1213]' \quad (52)$$

and

$$\mathbf{q}'_1 = [2.7071 \quad 1.2929 \quad -0.4142]', \mathbf{q}'_2 = [0.7929 \quad 2.2071 \quad 0.2929]', \mathbf{q}'_3 = [3.7071 \quad 2.2929 \quad -1.8284]'. \quad (53)$$

Using these coordinates, the dual matrix that describes the three line-vectors directed from \mathbf{p}'_i to $\mathbf{q}'_i, i = 1, 2, 3$, can be obtained, which is, of course, the matrix given in Eq. (51).

Having the coordinates of the initial positions of the points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, and their respective positions after applying the dual rotation matrix A , we now consider the inverse problem of estimating the dual rotation matrix A from experimentally ‘measured’ positions, which are corrupted by measurement noise [53,65,66].

3.2. Inverse problem of estimating the rotation matrix A

The inverse problem deals with estimating the unknown dual rotation matrix A from experimental measurements that are made to determine the coordinates of the points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, before rotation, and measurements made to determine the coordinates $\mathbf{p}'_i, \mathbf{q}'_i, i = 1, 2, 3$, of the respective locations of these points after the rotation A . These measured coordinates obtained before and after the rotation are the ‘raw measured data’ from which the dual rotation matrix A is to be estimated.

As mentioned before, these observed coordinates have measurement errors. First, from these measured coordinates, the experimentally obtained three line-vectors on the surface of the flat solar panel are found, both before and after the rotation A . From these line vectors the dual matrices P^m and P'^m before and after rotation are constructed (the superscript m stands for measured). The inverse problem of estimating the dual rotation matrix A is then handled by using these dual matrices P^m and P'^m that are constructed from the measured raw data.

To simulate the measurement errors, each measured coordinate is assumed to have a measurement error that is an independent random variable, which is uniformly distributed with zero mean lying in the interval $\eta \times \frac{1}{2}[-1, 1]$. The parameter η is a measure of the standard deviation of the measurement errors, and therefore reflects the accuracy of the measurements. One could choose, as well, a Gaussian error distribution or some other distribution, if additional data on measurement-error statistics is available. For simplicity, our results here are illustrated using a uniform distribution, so that the standard deviation of the measurement errors is $\eta/\sqrt{12}$.

A typical simulation of such measurement errors for $\eta = 10^{-4}$, gives a typical simulated set of what we consider ‘measured’ coordinates. Thus, we have

$$\begin{aligned} \mathbf{p}_1^m &:= \mathbf{p}_1 + \eta\gamma_1 = \mathbf{p}_1 + 10^{-4} [0.3143, \quad -0.1500, \quad 0.1160]', \\ \mathbf{p}_2^m &:= \mathbf{p}_2 + \eta\gamma_2 = \mathbf{p}_2 + 10^{-4} [-0.2565, \quad -0.3034, \quad -0.0267]', \\ \mathbf{p}_3^m &:= \mathbf{p}_3 + \eta\gamma_3 = \mathbf{p}_3 + 10^{-4} [0.4293, \quad -0.2489, \quad -0.1483]', \end{aligned} \quad (54)$$

and

$$\begin{aligned} \mathbf{q}_1^m &:= \mathbf{q}_1 + \eta\gamma_4 = \mathbf{q}_1 + 10^{-4} [0.3308, \quad 0.4172, \quad 0.2537]', \\ \mathbf{q}_2^m &:= \mathbf{q}_2 + \eta\gamma_5 = \mathbf{q}_2 + 10^{-4} [0.0853, \quad -0.2142, \quad -0.1196]', \\ \mathbf{q}_3^m &:= \mathbf{q}_3 + \eta\gamma_6 = \mathbf{q}_3 + 10^{-4} [0.0497, \quad 0.2572, \quad 0.0678]', \end{aligned} \quad (55)$$

where the ‘exact’ coordinates of the points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, are given in Eqs. (44) and (45), respectively. The second members on the right-hand sides in Eqs. (54) and (55) represent random measurement errors, which are displayed here to only 4 decimal places.

Using Eqs. (54) and (55), the three vector-lines from \mathbf{p}_i^m to $\mathbf{q}_i^m, i = 1, 2, 3$, are used to construct the ‘measured’ dual matrix of line-vectors, P^m , given by

$$P^m = \begin{bmatrix} 1.0000 & -1.0000 & 1.0000 \\ 0.0001 & 0.0000 & 0.0001 \\ 1.0000 & -3.0000 & 3.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.9999 & -2.9999 & 3.0000 \\ -1.0000 & 6.0001 & -13.0002 \\ -0.9999 & 1.0000 & -0.9997 \end{bmatrix}. \quad (56)$$

The elements of P^m are shown correct to only 4 decimal places.

Similarly, the coordinates of the points after rotation of 90° about the unit vector $\mathbf{v}_0 = [\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0]^T$ passing through the point $w_0 = (0, \quad 0, \quad 1)$ are measured. Using a simulated typical set of measurement errors that are random, have zero mean, are independent, and uniformly distributed again lying in the interval $\eta \times \frac{1}{2}[-1, 1]$ with $\eta = 10^{-4}$, we obtain the ‘measured’ positions of the respective points after rotation as

$$\begin{aligned} \mathbf{p}'_1{}^m &:= \mathbf{p}'_1 + \eta\gamma'_1 = \mathbf{p}'_1 + 10^{-4} [-0.3810, \quad -0.1596, \quad 0.2513]', \\ \mathbf{p}'_2{}^m &:= \mathbf{p}'_2 + \eta\gamma'_2 = \mathbf{p}'_2 + 10^{-4} [-0.0016, \quad 0.0853, \quad -0.2449]', \\ \mathbf{p}'_3{}^m &:= \mathbf{p}'_3 + \eta\gamma'_3 = \mathbf{p}'_3 + 10^{-4} [0.4597, \quad -0.2762, \quad 0.0060]', \end{aligned} \quad (57)$$

and

$$\begin{aligned} \mathbf{q}'_1^m &:= \mathbf{q}'_1 + \eta\gamma'_4 = \mathbf{q}'_1 + 10^{-4} [0.1991, \quad 0.0472, \quad -0.2425]', \\ \mathbf{q}'_2^m &:= \mathbf{q}'_2 + \eta\gamma'_5 = \mathbf{q}'_2 + 10^{-4} [0.3909, \quad -0.3614, \quad 0.3407]', \\ \mathbf{q}'_3^m &:= \mathbf{q}'_3 + \eta\gamma'_6 = \mathbf{q}'_3 + 10^{-4} [0.4593, \quad -0.3507, \quad -0.2457]'. \end{aligned} \tag{58}$$

Here, the ‘exact’ coordinates of the points \mathbf{p}'_i and \mathbf{q}'_i , $i = 1, 2, 3$, are given in Eqs. (52) and (53). The second members in Eqs. (57) and (58), as before, are the contributions of random measurement errors, yielding the ‘measured’ coordinates.

From these measured positions, the three line-vectors from \mathbf{p}'_i to \mathbf{q}'_i , $i = 1, 2, 3$, can be obtained, and hence the dual matrix of the three line-vectors is constructed to yield

$$P^m = \begin{bmatrix} 1.2072 & -2.6213 & 2.6213 \\ -0.2071 & 1.6213 & -1.6213 \\ -0.7072 & 0.7072 & -0.7071 \end{bmatrix} + \varepsilon \begin{bmatrix} -1.0001 & 1.0858 & -4.5859 \\ 1.4143 & -1.3286 & -2.1715 \\ -2.1213 & 7.0709 & -12.0208 \end{bmatrix}. \tag{59}$$

Having constructed the dual matrices P^m and P'^m using the raw measured data, we now begin the inverse problem of estimating the dual rotation matrix A , whose actual value is given in Eq. (49).

This inverse problem can be solved by observing that the transpose of Eq. (50) yields

$$UA^T = U' \tag{60}$$

where $U = P^T$ and $U' = (P')^T$, and A^T is the transpose of the dual rotation matrix A .

Assuming that the Moore–Penrose inverse, U^+ , of the matrix U exists, an estimate, A_{est} , of the rotation matrix A can be obtained by solving the linear Eq. (60), as

$$A_{est}^T = U^+U'. \tag{61}$$

But the true dual matrices P (and therefore, $U = P^T$) and P' are not available because of measurement errors; only the dual matrix P^m constructed from measurements (and therefore $U^m = (P^m)^T$) and the dual matrix P'^m , also constructed from measurements (and therefore $U'^m = (P'^m)^T$) are known Eqs. (56) and ((59)). Thus we use Eq. (60), after replacing U by U^m , and U' by U'^m , to get

$$A_{est}^T = (U^m)^+U'^m, \tag{62}$$

which gives an estimate of the dual matrix A . One would expect that this procedure should work reasonably well², in the same way that it does for ordinary matrices, except that the dual matrix $U^m = (P^m)^T$ does not have a Moore–Penrose dual generalized inverse since the necessary and sufficient conditions given in Result 2 are not met for the matrix P^m , and therefore for the matrix U^m (see Corollary 3). Therefore Eq. (62) cannot be used! Naively generated computational results will thus produce erroneous answers.

However, with a set of observation points $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, on a rigid solar panel that is not flat, consider the six representative points on it that have coordinates

$$\begin{aligned} \mathbf{p}_1 &= [2 \quad 1 \quad 1]', \quad \mathbf{p}_2 = [3 \quad 1.5 \quad 3]', \quad \mathbf{p}_3 = [4 \quad 1 \quad -1]', \quad \text{and} \\ \mathbf{q}_1 &= [3 \quad 1.25 \quad 2]', \quad \mathbf{q}_2 = [2 \quad 1 \quad 0]', \quad \mathbf{q}_3 = [5 \quad 1 \quad 2]'. \end{aligned} \tag{63}$$

The dual line-vectors from $\mathbf{p}_i, \mathbf{q}_i, i = 1, 2, 3$, yield the dual matrix

$$P = \begin{bmatrix} 1.0000 & -1.0000 & 1.0000 \\ 0.2500 & -0.5000 & 0 \\ 1.0000 & -3.0000 & 3.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.7500 & -3.0000 & 3.0000 \\ -1.0000 & 6.0000 & -13.0000 \\ -0.5000 & 0 & -1.0000 \end{bmatrix}, \tag{64}$$

which satisfies the conditions given in Result 2, and therefore has an MPDGI. By Corollary 3, P^T also has an MPDGI then.

After a rotation of 90° about the unit vector \mathbf{v}_0 passing through the point w_0 , as before, these points occupy the positions, respectively,

$$\begin{aligned} \mathbf{p}'_1 &= [1.5 \quad 1.5 \quad 0.2929]', \quad \mathbf{p}'_2 = [3.6642 \quad 0.8358 \quad -0.0607]', \quad \mathbf{p}'_3 = [1.08 \quad 3.9142 \quad -1.1213]', \quad \text{and,} \\ \mathbf{q}'_1 &= [2.8321 \quad 1.4179 \quad -0.2374]', \quad \mathbf{q}'_2 = [0.7929 \quad 2.2071 \quad 2.2071]', \quad \mathbf{q}'_3 = [3.7071 \quad 2.2929 \quad -1.8284]'. \end{aligned} \tag{65}$$

² The connection between the dual Moore–Penrose inverse and the ‘least-squares’ approach is an active area of current research.

The noise corrupted measured positions, of the points $\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}'_i, \mathbf{q}'_i, i = 1, 2, 3$, are taken to be

$$\mathbf{p}_i^m = \mathbf{p}_i + \bar{\eta}\gamma_i, \mathbf{q}_i^m = \mathbf{q}_i + \bar{\eta}\gamma_{i+3}, i = 1, 2, 3, \tag{66}$$

$$\mathbf{p}'_i{}^m = \mathbf{p}'_i + \bar{\eta}\gamma'_i, \mathbf{q}'_i{}^m = \mathbf{q}'_i + \bar{\eta}\gamma'_{i+3}, i = 1, 2, 3, \tag{67}$$

in which the ‘exact’ positions of the points $\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}'_i, \mathbf{q}'_i, i = 1, 2, 3$, are given in Eqs. (63) and (65), respectively; the values of $\gamma_i, \gamma'_i, i = 1, 2, 3, 4, 5, 6$, are the same as those in Eqs. (54), (55), (57) and (58). We consider here the measurement noise to be 10 times larger than before, and so $\bar{\eta} = 10^{-3}$.

Using the ‘measured’ coordinates of the points given by Eqs. (66) and (67)—our raw measured data—the dual matrices corresponding to the three line-vectors from \mathbf{p}_i^m to $\mathbf{q}_i^m, i = 1, 2, 3$, before and after the same rotation (of 90° about the unit vector $\mathbf{v}_0 = [\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} 0]^T$ passing through the point $w_0 = (0, 0, 1)$) are constructed. They are, respectively,

$$P^m = \begin{bmatrix} 1.0000 & -0.9997 & 0.9996 \\ 0.2506 & -0.4999 & 0.0005 \\ 1.0001 & -3.0001 & 3.0002 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.7494 & -2.9995 & 3.0000 \\ -1.0005 & 6.0006 & -13.0019 \\ -0.4987 & -0.0004 & -0.9973 \end{bmatrix} \tag{68}$$

and

$$P'^m = \begin{bmatrix} 1.3327 & -2.8709 & 2.6213 \\ -0.0819 & 1.3709 & -1.6214 \\ -0.5308 & 0.3541 & -0.7074 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.7721 & 0.3795 & -4.5867 \\ 1.1867 & -1.1228 & -2.1710 \\ -2.1216 & 7.4229 & -12.0209 \end{bmatrix}. \tag{69}$$

It is important to note that now the Moore–Penrose generalized inverse of the matrix $U^m = (P^m)^T$ exists, and using Eq. (62) the estimate of the dual rotation matrix A is computed to be

$$A_{est} = \begin{bmatrix} 0.5004 & 0.4999 & 0.7069 \\ 0.4995 & 0.5014 & -0.7069 \\ -0.7081 & 0.7073 & 0.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} -1.0017 & 0.0037 & 0.7062 \\ 0.0021 & 0.9786 & 0.7141 \\ -0.7082 & -0.7101 & 0.0028 \end{bmatrix}. \tag{70}$$

The error in this estimate from its exact value, A (see Eq. (49)), namely, $\Delta_{est} = A_{est} - A$, is then

$$\Delta_{est} = \begin{bmatrix} 0.0004 & -0.0001 & -0.0002 \\ -0.0005 & 0.0014 & 0.0002 \\ -0.0010 & 0.0002 & 0.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.0017 & 0.0037 & -0.0009 \\ 0.0021 & -0.0214 & 0.0070 \\ -0.0011 & -0.0030 & 0.0028 \end{bmatrix}. \tag{71}$$

If the accuracy of our measurements in locating the coordinates of the points is increased by a factor of 10, so that $\bar{\eta} = 10^{-4}$ leaving all the other parameters unchanged, the estimate of the rotation matrix now becomes

$$A_{est} = \begin{bmatrix} 0.5000 & 0.5000 & 0.7071 \\ 0.4999 & 0.5001 & -0.7071 \\ -0.7072 & 0.7071 & 0.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} -1.0002 & 0.0004 & 0.7070 \\ 0.0002 & 0.9979 & 0.7078 \\ -0.7072 & -0.7074 & 0.0003 \end{bmatrix}. \tag{72}$$

The error of this new estimate from the exact dual rotation matrix, A , is now

$$\Delta_{est} = 10^{-3} \begin{bmatrix} 0.0414 & -0.0052 & -0.0210 \\ -0.0524 & 0.1426 & 0.0180 \\ -0.0971 & 0.0228 & 0.0031 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.0002 & 0.0004 & -0.0001 \\ 0.0002 & -0.0021 & 0.0007 \\ -0.0001 & -0.0003 & 0.0003 \end{bmatrix}. \tag{73}$$

When compared with Eq. (71) and (73) shows a reduction in the estimation error of A of approximately an order of magnitude.

Remark 3. One might intuitively think that our estimates of the rotation matrix A for the case of the flat solar panel can be improved by using more precise measurements—our raw measured data—from which the matrices P^m and P'^m Eqs. (56) and ((59)) are constructed for use in Eq. (62). It is indeed true that, in general, measurements made with higher precision improve the quality of estimates obtained from Eq. (62) However, using a higher precision here will not help. In fact, as the precision of measurements is increased, the matrix P^m in Eq. (56) tends to the matrix P in Eq. (46) i.e., as $\eta \rightarrow 0$, the measured quantities $\mathbf{p}_i^m, \mathbf{q}_i^m \rightarrow \mathbf{p}_i, \mathbf{q}_i$ for $1 \leq i \leq 3$, and therefore $P^m \rightarrow P$. But we have shown that P and P^T (by Corollary 3) do not have Moore–Penrose dual generalized inverses! Hence some-what surprisingly, increased precision cannot, generically speaking, provide better estimates of the dual matrix A . □

Remark 4. It is important to note that the breakdown in the estimation of the dual rotation matrix A in the case of the flat solar panel is related to the non-existence of an MPDGI of the matrix P^m , and therefore of $U^m = (P^m)^T$; it has nothing to do with the actual rotation matrix A . We would thus not be able to estimate any (every) rotation matrix A from such measured data, and estimates of the rotation matrix would not be possible through the use of Eq. (62).

On the other hand, in the case of a non-flat solar panel as shown by Eqs. (71) and (73), when the MPDGI of P^m exists, by increasing the precision with which the coordinates of the points on the panel’s surface are measured, and therefore by increasing the precision with which the matrices P^m and P'^m are constructed, progressively improved estimates of the exact dual rotation matrix, A , are obtained. □

Remark 5. A perceptive experimentalist might use the knowledge that the flat solar panel lies in a plane parallel to the x - z plane, and therefore the measured y -coordinates of all the points \mathbf{p}_i , \mathbf{q}_i , $i = 1, 2, 3$, (see Eqs. (44) and (45)) before rotation should, in fact, be identical. This then results in the recognition that: (1) the real part of the matrix P^m in Eq. (56) should always have its second row to be zero, and (2) the elements of the second row in this matrix differ from zero only because of the presence of measurement errors. It then appears reasonable to replace the second row of the real part of matrix P^m in Eq. (56) and set it to zero, the thinking being that the errors in estimating the rotation matrix A would be reduced by reducing the deleterious effect of measurement errors in the elements of this second row, which are known to be exactly zero because the panel lies in the x - z plane. However, despite its plausibility, what actually results is quite the contrary. For now the real part of the matrix P^m has the structure shown in Eq. (38) which almost always has no Moore–Penrose dual generalized inverse! And no increase (or decrease) in the precision of measurements of the coordinates, which now affect only the entries in the first and third row of the real part of P^m since its second row is set to zero, will change that. Significant errors in the estimates of A would then generically (almost always) result when using Eq. (62). □

4. Conclusions

A constructive method has been developed to show that not all dual matrices have {1,2,3,4}-dual generalized inverses (MPDGI). There are an uncountably infinite number of dual matrices that have no MPDGIs. The necessary and sufficient conditions for a matrix to be the {1,2,3,4}-generalized inverse (MPDGI), or some other generalized inverse like a {1,2}-dual generalized inverse, of a dual matrix are obtained. A simple necessary condition, which can be easily computationally checked, is also provided for a matrix to be an MPDGI of a given dual matrix.

The fact that all dual matrices do not have generalized inverses, has important practical implications, since dual matrices are nowadays widely used in many areas of science and engineering. The example provided herein from the area of kinematics in which dual matrices are often used shows that physically meaningful dual matrices can, and do, arise in the description of mechanical systems that do not have generalized inverses. Generalized inverses are commonly used when dealing with inverse problems, and the pitfalls and some-what non-intuitive results obtained in the solution of such problems when dual matrices do not have Moore–Penrose inverses are illustrated. Therefore, dual matrices need to be carefully vetted before attempting to find their generalized inverses, especially if they are generated from observed (or simulated) data.

More generally, this paper shows that mathematical formulations and/or computations that use generalized inverses of dual matrices in science and engineering must be handled with considerable care and caution because, unlike generalized inverses of ordinary matrices, generalized inverses of dual matrices are not guaranteed to exist.

Declaration of Competing Interest

None.

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